Math 247A Lecture 19 Notes

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1 The Mikhlin Multiplier Theorem and Properties of Littlewood-Paley Projections

1.1 The Mikhlin multiplier theorem

Theorem 1.1 (Mikhlin multiplier theorem)**.** *Let $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be such that $|D_\xi^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ uniformly for $|\xi| \neq 0$ and $0 \leq |\alpha| \leq \lceil \frac{d+1}{2} \rceil$. Then*

$$f \mapsto [m(\xi)\widehat{f}(\xi)]^\vee = m^\vee * f$$

is bounded on L^p for all $1 < p < \infty$.

Proof. By Plancherel and $m \in L^\infty$, we get boundedness on L^2. So it suffices to check regularity condition (c):

$$\int_{|x| \geq 2|y|} |m^\vee(x+y) - m^\vee(x)| \, dx \lesssim 1$$

uniformly in y. We have

$$\int_{|x| \geq 2|y|} |m^\vee(x+y) - m^\vee(x)| \, dx \lesssim \sum_{N \in 2^{\mathbb{Z}}} \int_{|x| \geq 2|y|} |m_N^\vee(x+y) - m_N^\vee(x)| \, dx$$

where $M_N = m\psi_N = m\psi(\,\cdot\,/N)$.

$$\begin{aligned}
&\leq \sum_{N \leq |y|^{-1}} \int_{|x|>2|y|} |m_N^\vee(x-y) - m_N^\vee(x)| \, dx \\
&\quad + 2 \sum_{N > |y|^{-1}} \int_{|x| \geq |y|} |m_N^\vee(x)| \, dx \\
&\leq \sum_{N \leq |y|^{-1}} \int_{|x| \geq 2|y|} |y| \int_0^1 |\nabla m_N^\vee(x + \theta y)| \, d\theta \, dx
\end{aligned}$$

$$+ 2 \sum_{N > |y|^{-1}} \int_{|x| \geq |y|} |m_N^\vee(x)| \, dx.$$

Last time, we had pointwise bound on derivatives by assuming more conditions for more values of α. Here, instead, we will use Plancherel. By Plancherel,

$$\begin{aligned}
\|(2\pi i x)^\alpha m_N^\vee(x)\|_{L^2_x} &= \|D_\xi^\alpha m_N\|_{L^2_\xi} \\
&= \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1, \alpha_2} \|D_\xi^{\alpha_1} m(\xi) \cdot \frac{1}{N^{|\alpha_2|}} (D_\xi^{\alpha_2}(\xi/N)\|_2 \\
&\lesssim_\alpha \sum_{\alpha_1 + \alpha_2} \left(\int |\xi|^{-2|\alpha_1|} N^{-2|\alpha_2|} \, d\xi \right)^{1/2} \\
&\lesssim N^{d/2 - |\alpha|}
\end{aligned}$$

for all $0 \leq \alpha \leq \lceil \frac{d+1}{2} \rceil$. By Cauchy-Schwarz,

$$\int_{|x| \leq A} |m_N^\vee(x)| \, dx \leq \|m_N^\vee\|_2 A^{d/2} \lesssim (AN)^{d/2}.$$

Similarly,

$$\begin{aligned}
\int_{|x| > A} |m_N^\vee(x)| \, dx &\leq \|x^\alpha m_N^\vee\|_2 \left(\int_{|x| > A} |x|^{-2|\alpha|} \, dx \right)^{1/2} \\
&\lesssim N^{d/2 - |\alpha|} A^{d/2 - |\alpha|},
\end{aligned}$$

provided $|\alpha| > d/2$. So for $\alpha = \lceil \frac{d+1}{2} \rceil > d/2$, we get

$$\int_{|x| > A} |m_N^\vee(x)| \, dx \lesssim (NA)^{d/2 - \lceil (d+1)/2 \rceil}.$$

Then

$$\sum_{|x| > |y|^{-1}} \int_{|x| \geq |y|} |m_N^\vee(x)| \, dx \lesssim \sum_{N > |y|^{-1}} (N|y|)^{d/2 - \lceil (d+1)/2 \rceil}$$

This is a geometric series, so it is smaller than a constant times its largest term.

$$\lesssim 1,$$

uniformly in $y \in \mathbb{R}^d$. Taking $A = N^{-1}$ in our relations, we get

$$\int |m_N^\vee(x)| \lesssim 1,$$

uniformly in N.

The same arguments would give

$$\int |\nabla m_N^\vee(x)|\,dx \lesssim N,$$

uniformly in N. Indeed,

$$\begin{aligned}
\|(2\pi i x)^\alpha \nabla m_N^\vee\|_2 &= \|D^\alpha(\xi m_N)\|_2 \\
&\lesssim_\alpha \sum_{\alpha_1+\alpha_2=\alpha} \left(\int_{|\xi|\sim N} |\xi|^{2-2|\alpha_1|} N^{-2|\alpha_2|}\,d\xi \right)^{1/2} \\
&\lesssim N^{1+d/2-|\alpha|},
\end{aligned}$$

so we get

$$\int_{|x|\le A} |\nabla| m_N^\vee| \lesssim N^{1+d/2} A^{d/2} = n(NA)^{d/2},$$

$$\int_{|x|\ge A} |\nabla m_N^\vee| \lesssim N(NA)^{d/2-\lceil (d+1)/2\rceil}.$$

We can now estimate

$$\sum_{N\le |y|^{-1}} |y| \int_{|x|\ge 2|y|} \int_0^1 |\nabla m_N^\vee(x+\theta y)|\,d\theta\,dx \lesssim \sum_{N\le |y|^{-1}} |y|\cdot N \lesssim 1,$$

uniformly in y. □

1.2 Properties of Littlewood-Paley projections

Recall the Littlewood-Paley projections:

$$\varphi(x) = \begin{cases} 1 & |x| \le 1.4 \\ 0 & |x| > 1.42, \end{cases} \qquad \psi(x) = \varphi(x) - \varphi(2x).$$

Then we had

$$f_N = P_N f = f * N^d \psi^\vee(N\,\cdot),$$

$$f_{\le N} = P_{\le N} f = f * N^d \varphi^\vee(N\,\cdot).$$

Here are the basic properties of Littlewood-Paley projections.

Theorem 1.2.

1. $\|f_n\|_p + \|f_{\le N}\|_p \lesssim \|f\|_p$ *uniformly in* N *and for* $1 \le p \le \infty$.

2. $|f_N(x)| + |f_{\leq N}(x)| \lesssim Mf(x)$.

3. *For* $f \in L^p$ *with* $1 < p < \infty$, *we have* $f \overset{L^p}{=} \sum_{N \in 2^{\mathbb{Z}}} f_N$.

4. *(Bernstein's inequality) For* $1 \leq p \leq q \leq \infty$,

$$\|f_N\|_q \lesssim N^{d/p - d/q} \|f_N\|_p$$

$$\|f_{\leq N}\|_q \lesssim N^{d/p - d/q} \|f_{\leq N}\|_p.$$

5. *(Bernstein) For* $1 \leq p \leq \infty$ *and* $s \in \mathbb{R}$,

$$\||\nabla|^s f_N\|_p \sim N^s \|f_N\|_p.$$

In particular, for $s > 0$ *and* $1 \leq p \leq \infty$,

$$\||\nabla|^s f_{\leq N}\|_p \lesssim N^s \|f_{\leq N}\|_p.$$

$$\|\nabla|^{-s} f_{>N}\|_p \lesssim N^{-s} \|f_{>N}\|_p.$$

Proof.

1. By Young's inequality,

$$\begin{aligned} \|f_N\|_p &= \|f * N^d \psi^\vee(N\,\cdot)\|_p \\ &\lesssim \|f\|_p \underbrace{\|N^d \psi^\vee(N\,\cdot)\|_1}_{} = \|\psi^\vee\|_1 \\ &\lesssim \|f\|_p, \end{aligned}$$

$$\begin{aligned} \|f_{\leq N}\|_p &= \|f * N^d \varphi^\vee(N\,\cdot)\|_p \\ &\lesssim \|f\|_p \|\varphi_1^\vee\|_1 \\ &\lesssim \|f\|_p. \end{aligned}$$

2.

$$\begin{aligned} |f_N(x)| &\leq \int |f(y) N^d \psi^\vee(N(x-y))|\,dy \\ &\lesssim N^d \int |f(y)| \frac{1}{\langle N(x-y)\rangle^{2d}}\,dy \\ &\lesssim N^d \int_{|x-y| \leq 1/N} |f(y)|\,dy + \sum_{R \in 2^{\mathbb{Z}}} N^d \int_{R/N \leq |x-y| \leq 2R/N} \frac{|f(y)|}{R^{2d}}\,dy \end{aligned}$$

How do we make this look like the maximal function?

$$\begin{aligned}
&\lesssim \frac{1}{|B(x,1/N)|}\int_{B(x,1/N)} |f(y)|\,dy \\
&\quad + \sum_{R\in 2^{\mathbb{Z}}} R^{-d}\frac{1}{|B(x,2R/N)|}\int_{B(x,2R/N)} |f(y)|\,dy \\
&\lesssim Mf(x)\left[1+\sum_{R\in 2^{\mathbb{Z}}} R^{-d}\right] \\
&\lesssim Mf(x).
\end{aligned}$$

3. First assume $f \in \mathcal{S}(\mathbb{R}^d)$. By Plancherel and dominated convergence,

$$\|f - P_{N\le\cdot\le 1/N} f\|_2 \xrightarrow{N\to 0} 0.$$

For $1 < p < 2$, write $\frac{1}{p} = \theta + \frac{1-\theta}{2} = \frac{1+\theta}{2}$.

$$\begin{aligned}
\|f - P_{N\le\cdot\le 1/N} f\|_p &\le \|f - P_{N\le\cdot\le 1/N} f\|_1^{\theta}\|f - P_{N\le\cdot\le 1/N} f\|_2^{1-\theta} \\
&\le (\|f\|_1 + \|P_{N\le\cdot\le 1/N} f\|_1)^{\theta}\cdot\|f - P_{N\le\cdot\le 1/N} f\|_2^{1-\theta} \\
&\xrightarrow{N\to 0} 0
\end{aligned}$$

by property (1). For $2 < p < \infty$,

$$\|f - P_{N\le\cdot\le 1/N} f\|_p \le \underbrace{\|\quad\|_2^{2/p}}_{\xrightarrow{N\to 0} 0}\underbrace{\|\quad\|_\infty^{1-2/p}}_{\lesssim \|f\|_\infty}$$

If $f \in L^p$, let $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - g\|_p \le \delta$. Then

$$\begin{aligned}
\|f - P_{N\le\cdot\le 1/N} f\|_p &\lesssim \|g - P_{N\le\cdot\le 1/N} g\|_p + \|f-g\||_p + \|P_{N\le\cdot\le 1/N}(f-g)\|_p \\
&\lesssim o(1) + \delta
\end{aligned}$$

as $N \to 0$. □

We will prove (4) and (5) next time.

Remark 1.1. (3) fails for $p = 1$ and $p = \infty$. For $p = 1$, $\int P_N f = \widehat{P_n f}(0) = 0$, so pick some function with mean 0.